

Ray Singer Analytic Torsion of CY Manifolds II.

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ABSTRACT. In this paper we construct the analogue of Dedekind eta function for odd dimensional CY manifolds. We use the theory of determinant line bundles. We constructed a canonical holomorphic section η^N of some power of the determinant line bundle on the moduli space of odd dimensional CY manifolds.

According to Viehweg the moduli space of moduli space of polarized odd dimensional CY manifolds $\mathcal{M}(M)$ is quasi projective. According to a Theorem due to Hironaka we can find a projective smooth variety $\overline{\mathcal{M}}(M)$ such that $\overline{\mathcal{M}}(M) \setminus \mathcal{M}(M) = \mathcal{D}_\infty$ is a divisor of normal crossings. We also showed by using Mumford's theory of metrics with logarithmic growths that the determinant line bundle can be canonically prolonged to $\overline{\mathcal{M}}(M)$. We also showed that there exists section η of some power of the determinant line bundle which vanishes on \mathcal{D}_∞ and has a Quillen norm the Ray Singer Analytic Torsion. This section is that analogue of the Dedekind eta Function η .

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1. INTRODUCTION.

1.1. Analytic Torsion of Elliptic Curves and Kronecker's Limit Formula. In case of elliptic curves the exponential of Ray Singer analytic torsion is the Quillen norm of a non vanishing section of the determinant line bundle. Kronecker limit formula states that the Ray Singer analytic torsion is exactly the Quillen norm of the Dedekind eta function η . One of the versions of the proof of the Kronecker limit formula is based on the facts that log of the regularized determinant of the flat metric on the elliptic curve is the potential for the Poincare metrics and this determinant is bounded as a function on the moduli space. For this proof of the Kronecker limit formula see [18].

Kronecker limit formula can be interpreted as the existence of a section η^{24} of some power of the determinant line bundle whose Quillen norm is exactly the 24th power of the Ray Singer Analytic torsion.

1.2. Formulation of the Problem Discussed in the Paper.

Problem 1. Does there exists a section η^N of some power of the determinant line bundle over the moduli space of odd dimensional CY manifolds whose Quillen norm is the N^{th} power of the Ray Singer Analytic torsion?

The positive answer to Problem 1 can be considered as a generalization of the above mentioned relations between Ray Singer Analytic torsion and the Quillen norm on the determinant line bundle from elliptic curves to odd dimensional Calabi-Yau manifolds. More precisely the positive answer of the Problem 1 will be a generalization of Kronecker's Limit Formula to higher dimensions.

1.3. Description of the Ideas Used in the Paper. The idea on which this paper is based, is very simple. The Quillen metric is related to the spectral properties of the Laplacians acting on $(0,q)$ forms in case of Kähler manifolds. The main question is when the Ray Singer analytic torsion is the Quillen metric of some holomorphic section of the determinant line bundle.

One of the main results of the paper is the construction of a canonical section of the determinant line bundle over the moduli space of odd dimensional CY manifolds up to a constant whose absolute value is one. First we prove that the determinant line bundle \mathcal{L} as C^∞ bundle is trivial. The proof that the determinant line bundle \mathcal{L} is a trivial C^∞ is based on the facts that for odd dimensional CY manifolds we have $\dim \ker(\bar{\partial} \bar{\partial}^*) = \dim \ker(\bar{\partial}^* \bar{\partial}) = 1$ over the moduli space and the Ray Singer Analytic Torsion $I(M)$ is not a constant and strictly positive function on the moduli space. The definition of the Quillen metric on the determinant line bundle \mathcal{L} implies that $c_1(\mathcal{L}) = d(\partial(\log(I(M))))$. Therefore $c_1(\mathcal{L})$ is an exact two form on $\mathcal{M}(M)$.

It is easy to see that the determinant line bundle is isomorphic to $\pi_*(\Omega_{\mathcal{X}/\mathcal{M}(M)}^{2n+1})$. This implies that we can construct a non zero section of the determinant line bundle whose L^2 norm pointwise is equal to one. Using that section we can construct a canonical C^∞ non vanishing section of the determinant line bundle up to a constant whose Quillen norm is exactly the analytic Ray Singer torsion. We will call such section $\det(\bar{\partial})$.

It is not difficult to see that the analytic torsion for even dimensional CY manifolds is equal to zero and for odd dimensional CY manifolds it is different from zero. On the other hand the index of the $\bar{\partial}$ operator on the complex of $(0,q)$ forms is zero for odd dimensional CY manifolds therefore there exists a canonical non vanishing section $\det(\bar{\partial})$ whose Quillen norm is exactly the Ray Singer Analytic Torsion.

We will study the zero set of the canonical section $\det(\bar{\partial})$ in this paper on some compactification of $\overline{\mathcal{M}'(M)}$ such that $\overline{\mathcal{M}'(M)} \setminus \mathcal{M}'(M)$ is a divisor of normal crossings. Viehweg proved in [30] that $\mathcal{M}'(M)$ is a quasi-projective variety. Moreover it is a well known fact that the moduli space of CY manifolds is obtained by factoring the Teichmüller space by subgroup of finite index in the mapping class group that preserves some polarization on the CY manifolds, which according to Sullivan is an arithmetic group. From the fact that the mapping class group is an arithmetic one can find a subgroup of finite index in the mapping class group such that the quotient of the Teichmüller space by this group is a non-singular variety $\mathcal{M}(M)$ which is a finite covering of $\mathcal{M}'(M)$.

Let $\overline{\mathcal{M}(M)}$ be some compactification of $\mathcal{M}(M)$ such that $\overline{\mathcal{M}(M)} \setminus \mathcal{M}(M) = \mathfrak{D}_\infty$ is a divisor with normal crossings. The divisor \mathfrak{D}_∞ will be called the discriminant locus. It is easy to see that the determinant line bundle \mathcal{L} for odd dimensional CY manifolds is isomorphic to the dual of the holomorphic line bundle $R^0 \pi_* \Omega^{2n+1}$. On the line bundle

$R^0\pi_*\Omega^{2n+1}$ we have a natural metric $\|\cdot\|^2$. One can show that the metric $\|\cdot\|^2$ has a logarithmic growth in the sense of Mumford. See [21]. From here we deduced that determinant line bundle \mathcal{L} can be prolonged in an unique way to a line bundle $\overline{\mathcal{L}}$ over $\overline{\mathcal{M}(M)}$ using the metric $\|\cdot\|^2$.

Since we proved in [28] that the Quillen norm of the canonical section $\det(\overline{\partial})$ is bounded, it is not difficult see that we can continue the C^∞ section $\det(\overline{\partial})$ to a section $\det(\overline{\partial})$ of the line bundle $\overline{\mathcal{L}}$ over $\overline{\mathcal{M}(M)}$ and whose zero set is supported by \mathcal{D}_∞ .

1.4. Formulation of the Main Result. The main theorem of the paper is the following one:

THEOREM. There exists a holomorphic section $\overline{\eta}^N$ of the line bundle $(\overline{\mathcal{L}})^{\otimes N}$ over $\overline{\mathcal{M}(M)}$ whose zero set is supported by \mathcal{D}_∞ and the Quillen norm of $\overline{\eta}^N|_{\mathcal{M}(M)} = I(M)^N$.

1.5. Outline of the Proof of the Main Result. The ideas of the proof are the following:

Step 1. We noticed in [28] that the analytic torsion is equal to the determinant of the Laplacians $\det(\Delta_\tau)$ of Calabi-Yau metrics acting on functions, whose imaginary part has a fixed cohomology class, namely the polarization class. By using the variational formulas from [28] & [26] and the facts that both $\log(\det(\Delta_\tau))$ and $\log(\|\omega_\tau\|^2)$ are potentials of the Weil Petersson metric, we can prove that locally we have the following formula $\det(\Delta_\tau) = \|\omega_\tau\|^2 |f|^2$, where f is a holomorphic function and $\|\omega_\tau\|^2$ is a family of L^2 norms of holomorphic n forms on the CY manifolds. We will prove that $\|\omega_\tau\|^2$ has a logarithmic growth as τ approaches \mathcal{D}_∞ . We proved in [29] that the Ray Singer Analytic Torsion is bounded by a constant. Combining those two facts we deduce that the canonical $C^\infty \det(\overline{\partial})$ vanishes on \mathcal{D}_∞ .

Step 2. Next step is to show that some power of the determinant line bundle \mathcal{L}^N is a trivial line bundle over $\mathcal{M}(M)$. This fact follows from the following five ingredients.

1. We know that as a C^∞ line bundle \mathcal{L} is trivial over $\mathcal{M}(M)$.
2. The second fact is that \mathcal{L} as a holomorphic line bundle is isomorphic to $R^0\pi_*(\Omega_{\mathcal{X}/\mathcal{M}(M)}^{2n+1,0})$. So \mathcal{L} is a subbundle of the flat vector bundle $R^n\pi_*\mathbb{C}$.
3. The pull back of $R^n\pi_*\mathbb{C}$ is a trivial vector bundle over the Teichmüller space $\mathcal{T}(M)$ we can conclude that the pull back of \mathcal{L}^* on $\mathcal{T}(M)$ is also a trivial holomorphic bundle.
4. Since $\mathcal{M}(M) \cong \mathcal{T}(M)/\Gamma$, where Γ is some arithmetic group of rank ≥ 2 , we conclude that the holomorphic line bundle \mathcal{L} is a flat line bundle defined by character of Γ .
5. A theorem of Kazhdan, which states that $\Gamma/[\Gamma, \Gamma]$ is finite implies that \mathcal{L}^N is a trivial holomorphic line bundle over $\mathcal{M}(M)$.

From these five ingredients we deduce the existence of a non vanishing holomorphic section η of the determinant line bundle $\overline{\mathcal{L}}$ over $\overline{\mathcal{M}(M)}$ whose zero set is supported by the discriminant locus \mathcal{D}_∞ .

1.6. The Organization of the Paper. This article is organized as follows.

In **Section 2** we will construct the Teichmüller space based on the local deformation theory of CY manifolds developed in [26].

In **Section 3** we will prove the existence of a subgroup Γ of finite index in the mapping class group of a CY manifold such that this subgroup acts freely on the Teichmüller space. We also will show that the quotient of the Teichmüller space by Γ is a non-singular one. So we will construct a finite covering $\mathcal{M}(M)$ of the moduli space of CY manifolds $\mathcal{M}'(M)$, which is a non-singular quasi-projective variety.

In **Section 4** we recall the theory of determinant line bundles of Mumford, Knudsen, Bismut, Donaldson, Gillet and Soulé, following the exposition of D. Freed. See [9] and [11]. In this section we will construct a non vanishing section $\det(D)$ of the determinant line bundle \mathcal{L} over the moduli space $\mathcal{M}(M)$ of a CY manifold M of any dimension.

In **Section 5** we prove that the determinant line bundle is a trivial C^∞ bundle over the moduli space $\mathcal{M}(M)$. We also construct a unique up to a constant ξ such that $|\xi| = 1$ C^∞ section $\det(\bar{\partial})$ of the determinant line bundle \mathcal{L} which has a Quillen norm equal to Ray Singer Analytic torsion $I(M)$.

In **Section 6** we review the theory of metrics with logarithmic singularities following Mumford's article [21]. We will show that the determinant line bundle is isomorphic to the line bundle of holomorphic n -forms on the moduli space $\mathcal{M}(M)$ and the natural L^2 metric on that bundle has logarithmic singularities.

In **Section 7** we will use the results of the previous sections to deduce that we can prolong the determinant line bundle to a line bundle on any compactification $\overline{\mathcal{M}(M)}$ such that $\overline{\mathcal{M}(M)} \setminus \mathcal{M}(M) = \mathfrak{D}_\infty$ is a divisor with normal crossings. By using the fact proved in [28] that the analytic Ray-Singer torsion is equal to the determinant of the Laplacian of CY metric, acting on functions, we deduce that we can prolong the canonical C^∞ section $\det(\bar{\partial})$ to any compactification $\overline{\mathcal{M}(M)}$ such that $\overline{\mathcal{M}(M)} \setminus \mathcal{M}(M) = \mathfrak{D}_\infty$ is a divisor with normal crossings and that the zero set of $\det(D)$ is supported exactly by the discriminant locus \mathfrak{D}_∞ . Based on a result of Kazhdan and Sullivan we prove that there exists a positive integer N such that as a holomorphic line bundle the determinant line bundle to power N is a trivial one over the moduli space $\mathcal{M}(M)$. Using this result we constructed a holomorphic section $\bar{\eta}^N$ of the determinant line bundle $(\mathcal{L})^{\otimes N}$ over $\overline{\mathcal{M}(M)}$ whose zero set is supported by \mathfrak{D}_∞ .

In **Section 8** we discuss some applications of the results of this paper and some conjectures.

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2. TEICHMÜLLER THEORY OF CY MANIFOLDS.

2.1. Some Definitions.

Definition 3. We will define the Teichmüller space $\mathcal{T}(M)$ of a CY manifold M as follows:

$$\mathcal{T}(M) := \mathcal{I}(M) / \text{Diff}_0(M),$$

where $\mathcal{I}(M) := \{\text{the set of all integrable complex structures on } M\}$ and $\text{Diff}_0(M)$ is the group of diffeomorphisms isotopic to identity. The action of the group $\text{Diff}(M_0)$ is

defined as follows; Let $\phi \in \text{Diff}_0(M)$ then ϕ acts on integrable complex structures on M by pull back, i.e. if $I \in C^\infty(M, \text{Hom}(T(M), T(M)))$, then we define $\phi(I_{\Phi(\tau)}) = \phi^*(I_{\Phi(\tau)})$.

We will call a pair $(M; \gamma_1, \dots, \gamma_{b_n})$ a marked CY manifold where M is a CY manifold and $\{\gamma_1, \dots, \gamma_{b_n}\}$ is a basis of $H_n(M, \mathbb{Z})/\text{Tor}$.

Remark 4. It is easy to see that if we choose a basis of $H_n(M, \mathbb{Z})/\text{Tor}$ in one of the fibres of the Kuranishi family $\mathcal{M} \rightarrow \mathcal{K}$ then all the fibres will be marked, since as a C^∞ manifold $\mathcal{X}_\mathcal{K} \cong M \times \mathcal{K}$.

2.2. The Construction of the Teichmüller Space. The construction of the Teichmüller space is based on the following lemma:

Lemma 5. Let $\pi : \mathcal{M} \rightarrow \mathcal{K}$ be the Kuranishi family of a CY manifold M . Let ϕ be a complex analytic automorphism of M such that ϕ acts as an identity on $H_n(M, \mathbb{Z})$, then ϕ acts trivially on \mathcal{K} .

PROOF: Remark 4 implies that we may suppose that the Kuranishi family is marked. So we can define the period map $p : \mathcal{K} \rightarrow \mathbf{P}(H^n(M, \mathbb{Z}) \otimes \mathbb{C})$ as follows:

$$p((M; \gamma_1, \dots, \gamma_{b_n})) = (\dots, \int_{\gamma_i} \omega_\tau, \dots) \in \mathbf{P}(H^n(M, \mathbb{Z}) \otimes \mathbb{C}),$$

where ω_τ is a family of holomorphic n -forms over \mathcal{K} . Local Torelli theorem states that p is a local isomorphism. See [13]. So we can assume that $\mathcal{K} \subset \mathbf{P}(H^n(M, \mathbb{Z}) \otimes \mathbb{C})$. From here Lemma 5 follows directly. ■.

Theorem 6. The Teichmüller space $\mathcal{T}(M)$ of CY manifold of dimension $n \geq 3$ exists and each connected component of $\mathcal{T}(M)$ is a complex analytic manifold of complex dimension $h^{n-1,1} = \dim_{\mathbb{C}} H^1(M, \Omega^{n-1})$.

PROOF OF THEOREM 6 . We will define $\mathcal{T}(M)$ as follows: Let $\mathfrak{T}(M)$ be the set of all marked Kuranishi families $\mathcal{X}_\mathcal{K} \rightarrow \mathcal{K}$, then $\mathcal{T}(M) := \mathfrak{T}(M)/\sim$, where \sim is the following equivalence relation. (Notice that the points of $\mathfrak{T}(M)$ are pairs $(\tau, \pi^{-1}(\tau))$, where τ is a point in some \mathcal{K} and $\pi^{-1}(\tau)$ is a marked CY manifolds.) We will say that $(\tau_1, \pi^{-1}(\tau_1)) \sim (\tau_2, \pi^{-1}(\tau_2))$ if and only if $\pi^{-1}(\tau_1)$ and $\pi^{-1}(\tau_2)$ are isomorphic as marked CY manifolds. Lemma 5 together with the result proved in [26] that the Kuranishi space is a non-singular complex analytic space of dimension $h^{2,1}$ implies that each component of $\mathcal{T}(M)$ is a complex analytic manifold of complex dimension $h^{n-1,1} = \dim_{\mathbb{C}} H^1(M, \Omega^{n-1})$. From the Mutsusaka-Mumford Theorem, we know that the moduli topology on \mathcal{K} is a Haussdorff one. [20]. Indeed the completeness of the Kuranishi family $\mathcal{X}_\mathcal{K} \rightarrow \mathcal{K}$ and the Theorem from [20] implies that if we have two families $\mathcal{Y}_\mathcal{K} \rightarrow \mathcal{K}$ and $\mathcal{X}_\mathcal{K} \rightarrow \mathcal{K}$ and sequence of points $\{\tau_i\}$ such that

$$\lim_{i \rightarrow \infty} \tau_i = \tau_0 \in \mathcal{K}$$

such that the fibres Y_{τ_i} and X_{τ_i} are isomorphic then X_{τ_0} is isomorphic to Y_{τ_0} . This fact is based on the observation that the isomorphism between Y_{τ_i} and X_{τ_i} should preserve a fixed polarization. We can apply the Bishop Theorem to conclude Theorem 6. See [4]. Theorem 6 is proved. ■.

In Theorem 6 we proved much more. Indeed it is easy to prove that if ϕ is a complex analytic automorphism of CY manifold M isotopic to identity and ϕ is of a finite order then it must be the identity. So the construction of $\mathcal{T}(M)$ implies directly that we constructed an universal family $\mathcal{M} \rightarrow \mathcal{T}(M)$ of marked CY manifolds.

From now on we will denote by $\mathcal{T}(M)$ the irreducible component of the Teichmüller space that contains our fix CY manifold M .

3. CONSTRUCTION OF THE MODULI SPACE OF POLARIZED CY MANIFOLDS

The group $\Gamma_1 := \text{Diff}^+(M)/\text{Diff}_0(M)$, where $\text{Diff}^+(M)$ is the group of diffeomorphisms preserving the orientation of M and $\text{Diff}_0(M)$ is the group of diffeomorphisms of M isotopic to identity will be called the mapping class group. A pair $(M; L)$ will be called a polarized CY manifold if $L \in H^2(M, \mathbb{Z})$ is a fixed class and there exists a Kähler metric G such that $[\text{Im}G] = L$. We will define $\Gamma_2 := \{\phi \in \Gamma_1 | \phi(L) = L\}$. The reason for using the group Γ_2 is that the moduli space $\Gamma_2 \backslash \mathcal{T}(M)$ will be Haussdorff.

Theorem 7. *There exists a subgroup of finite index Γ of Γ_2 such that Γ acts freely on $\mathcal{T}(M)$ and $\Gamma \backslash \mathcal{T}(M) = \mathfrak{M}(M)$ is a non-singular quasi-projective variety.*

PROOF OF THEOREM 7. The results of Sullivan from [24] imply that Γ_2 is an arithmetic group. This implies that in case of odd dimensional CY manifolds there is a homomorphism induced by the action of the diffeomorphism group on the middle homology with coefficients in $\mathbb{Z} : \phi : \Gamma_2 \rightarrow \text{Sp}(2b_n, \mathbb{Z})$ such that the image of Γ_2 has a finite index in the group $\text{Sp}(2b_n, \mathbb{Z})$ and $\ker(\phi)$ is a finite group.

In the case of even dimensional CY there is a homomorphism $\phi : \Gamma_2 \rightarrow \text{SO}(2p, q; \mathbb{Z})$ where $\text{SO}(2p, q; \mathbb{Z})$ is the group of the automorphisms of the lattice $H_n(M, \mathbb{Z})/\text{Tor}$, where the $\phi(\Gamma_2)$ has a finite index in the group $\text{SO}(2p, q; \mathbb{Z})$ and $\ker(\phi)$ is a finite group. A theorem of Borel implies that we can always find a subgroup of finite index Γ in Γ_2 such that Γ acts freely on $\text{Sp}(2b_n, \mathbb{R})/\text{U}(b_n)$ or on $\text{SO}_0(2p, q; \mathbb{R})/\text{SO}(2p) \times \text{SO}(q)$. We will prove that Γ acts without fixed point on $\mathcal{T}(M)$.

Suppose that there exists an element $g \in \Gamma$, such that $g(\tau) = \tau$ for some $\tau \in \mathcal{K} \subset \mathcal{T}(M)$. From local Torelli theorem we deduce that we may assume that the Kuranishi space \mathcal{K} is embedded in the \mathcal{G} , where \mathcal{G} is the classifying space of Hodge structures of weight n on $H^n(M, \mathbb{Z}) \otimes \mathbb{C}$. Griffiths proved in [13] that $\mathcal{G} \approx G/K$ where G in the odd dimensional case is $\text{Sp}(2b_n, \mathbb{R})$ and in the even dimensional is $\text{SO}_0(2p, q; \mathbb{R})$ and K is a compact subgroup of G .

Let K_0 is the maximal compact subgroup of G . So we have a natural C^∞ fibration $K_0/K \subset G/K \rightarrow G/K_0$. Griffith's transversality theorem implies that \mathcal{K} is transversal to the fibres K_0/K of the fibration $G/K \rightarrow G/K_0$.

The first part of our theorem follows from the fact that \mathcal{K} is transversal to the fibres K_0/K of the fibration $G/K \rightarrow G/K_0$ and the following observation; if $g \in \Gamma$ fixes a point $\tau \in G/K_0$, then $g \in K_0 \cap \Gamma$.¹ On the other hand side it is easy to see that local Torelli theorem implies that the action of Γ on \mathcal{K} is induced from the action Γ on G/K by left multiplications. From here we deduce that the action of Γ preserve the fibration $K_0/K \subset G/K \rightarrow G/K_0$. From here and the fact that Γ acts without fix point on G/K_0 the first part of our theorem follows directly. The second part of the theorem, namely that the space $\Gamma \backslash \mathcal{T}(M)$ is a quasi projective follows directly from the fact that $\Gamma \backslash \mathcal{T}(M) \rightarrow \Gamma_2 \backslash \mathcal{T}(M)$ is a finite map and that $\Gamma_2 \backslash \mathcal{T}(M)$ is a quasi projective variety according to [30]. Our theorem is proved. ■.

Remark 8. *From Theorem 7 it follows that we constructed a family of non-singular CY manifolds $\mathcal{X} \rightarrow \mathcal{M}(M)$ over a quasi-projective non-singular variety $\mathcal{M}(M)$. Moreover it is easy to see that $\mathcal{X} \subset \mathbb{CP}^N \times \mathcal{M}(M)$. So \mathcal{X} is also quasi-projective. From now on we will work only with this family.*

¹We suppose that K or K_0 acts on the right on G and Γ acts on the left on G .

4. THE THEORY OF DETERMINANT LINE BUNDLES

4.1. Geometric Data. In order to construct the determinant line bundle we need the following data:

1. A smooth fibration of manifolds $\pi : \mathcal{X} \rightarrow \mathcal{M}(M)$. In our case it will be the smooth fibration of the family of CY manifolds over the moduli space as defined in Theorem 7. Let $n = \dim_{\mathbb{C}} M$.
2. A metric along the fibres, that is a metric $g(\tau)$ on the relative tangent bundle $\mathcal{T}(\mathcal{X}/\mathcal{M}(M))$. In this paper the metric will be the families of CY metrics $g(\tau)$ such that the class of cohomology $[\text{Im}(g(\tau))] = L$ is fixed.

From these data we will construct the determinant line bundle \mathcal{L} over the moduli space of CY manifolds $\mathcal{M}(M)$. We will consider the relative $\bar{\partial}_{\mathcal{X}/\mathcal{M}(M)}$ complex:

$$0 \rightarrow \ker \bar{\partial}_{\mathcal{X}/\mathcal{M}(M)} \rightarrow C^\infty(\mathcal{X}) \xrightarrow{\bar{\partial}_{0,\mathcal{X}/\mathcal{M}(M)}} \Omega_{\mathcal{X}/\mathcal{M}(M)}^{0,1} \xrightarrow{\bar{\partial}_{1,\mathcal{X}/\mathcal{M}(M)}} \dots \Omega_{\mathcal{X}/\mathcal{M}(M)}^{0,n-1} \xrightarrow{\bar{\partial}_{n-1,\mathcal{X}/\mathcal{M}(M)}} \Omega_{\mathcal{X}/\mathcal{M}(M)}^{0,n} \rightarrow 0.$$

We will define D to be for each $\tau \in \mathcal{M}(M)$ and k ,

$$D_k := \bar{\partial}_{k,\mathcal{X}/\mathcal{M}(M)} + (\bar{\partial}_{k,\mathcal{X}/\mathcal{M}(M)})^* \quad \& \quad D_{k,\tau} := D_k|_{M_\tau} = \bar{\partial}_{k,\tau} + (\bar{\partial}_{k,\tau})^*.$$

Definition 9. We will call the above complex the relative Dolbault complex.

Let us define $(\mathcal{H}^k)_\tau := L^2(M_\tau, \Omega_\tau^{0,k})$. Furthermore, as τ varies over $\mathcal{M}(M)$, the spaces $(\mathcal{H}^k)_\tau$ fit together to form continuous Hilbert bundles \mathcal{H}^k over $\mathcal{M}(M)$.² Thus we can view $\bar{\partial}_{k,\mathcal{X}/\mathcal{M}(M)}$ as a bundle maps: $\bar{\partial}_{k,\mathcal{X}/\mathcal{M}(M)} : \mathcal{H}^k \rightarrow \mathcal{H}^{k+1}$. The Hilbert bundles \mathcal{H}^k carry L^2 metrics by definition.

4.2. Construction of the Determinant Line Bundle \mathcal{L} .

Some Basic Definitions. Now we are ready to construct the Determinant line bundle \mathcal{L} of the operator $\bar{\partial}_{\mathcal{X}/\mathcal{M}(M)}$. We will recall some basic consequences of the ellipticity if D_τ . Each fibre \mathcal{H}_τ^k of the Hilbert bundles \mathcal{H}^k decomposes into direct sum of eigen spaces of non-negative Laplacians $D_k D_k^*$ and $D_k^* D_k$. The spectrums of these operators are discrete, and the nonzero eigen values $\{\lambda_{k,i}\}$ of $D_k^* D_k$ and $D_k D_k^*$ agree and D_k define a canonical isomorphisms between the corresponding eigen spaces.

Definition 10. 1. Let $\mathcal{U}_a : \{\tau \in \mathcal{M}(M) | a \notin \text{Spec}(D_k D_k^*) \text{ for } 0 \leq k \leq n \text{ and any } a > 0\}$. (\mathcal{U}_a are open sets in $\mathcal{M}(M)$ and they form an open covering of $\mathcal{M}(M)$ since the spectrum of $D_\tau^* D_\tau$ is discrete.) **2.** Let the fibres of \mathcal{H}_a^k be the vector subspaces in $\mathcal{H}_{\tau,a}^k$ spanned by eigen vectors with eigen values less than a over \mathcal{U}_a . Then we can define the complex of :

$$0 \rightarrow \Gamma(\mathcal{U}_a, \mathcal{O}_{\mathcal{U}_a}) \rightarrow \mathcal{H}_a^0 \xrightarrow{\bar{\partial}_{0,\mathcal{X}/\mathcal{M}(M)}} \dots \rightarrow \mathcal{H}_a^{n-1} \xrightarrow{\bar{\partial}_{n-1,\mathcal{X}/\mathcal{M}(M)}} \mathcal{H}_a^n \rightarrow \ker(D_{n-1} \circ D_n^*) \rightarrow 0.$$

If $b > a$ we set $\mathcal{H}_{a,b}^k := \mathcal{H}_b^k / \mathcal{H}_a^k$. The spaces \mathcal{H}_a^k form a smooth finite dimensional C^∞ bundles over an open set $\mathcal{U}^a \subset \mathcal{M}(M)$. For the proof of this fact see [1].

²These bundles are not smooth since the composition $L^2 \times C^\infty \rightarrow L^2$ is not differentiable map.

Construction of the Generating Sections $\det(D_a)$ over \mathcal{U}_a .

Definition 11. Let $\omega_1^k, \dots, \omega_{m_k}^k, \psi_1^k, \dots, \psi_{N_k}^k, \phi_1^k, \dots, \phi_{M_k}^k$ be an orthonormal basis in the trivial vector bundle \mathcal{H}_a^k over \mathcal{U}_a , where $D_k \omega_i^k = 0$, $\bar{\partial}_k^* (\bar{\partial}_k \psi_j^k) = \lambda_j^k \psi_j^k$, $\bar{\partial}_k (\bar{\partial}_k^* \phi_j^k) = \lambda_j^k \phi_j^k$, $\phi_j^k \in \text{Im } \bar{\partial}_{k-1, \mathcal{X}/\mathcal{M}(M)}$ and $\psi_j^k \in \text{Im } (\bar{\partial}_{k, \mathcal{X}/\mathcal{M}(M)}^*)$ for $1 \leq i \leq k$ and $0 < \lambda_j < a$ for $1 \leq j \leq N$. Let

$$\det(\bar{\partial}_{k,a}) = (\omega_1^k \wedge \dots \wedge \omega_{m_k}^k \wedge (\bar{\partial}_{k-1, \mathcal{X}/\mathcal{M}(M)} \psi_1^{k-1}) \wedge \dots \wedge (\bar{\partial}_{k-1, \mathcal{X}/\mathcal{M}(M)} \psi_{N_k}^{k-1}) \wedge \phi_1^k \wedge \dots \wedge \phi_{M_k}^k)^{(-1)^k}.$$

We will define the line bundle \mathcal{L} restricted on \mathcal{U}_a as follows:

$$\mathcal{L}^a := \mathcal{L}|_{\mathcal{U}_a} = \otimes_{k=0}^n \left(\wedge^{\dim \mathcal{H}_a^k} \mathcal{H}_a^k \right)^{(-1)^k}.$$

Definition 12. The generator $\det(\bar{\partial}_a)$ of $\mathcal{L}^a := \mathcal{L}|_{\mathcal{U}_a}$ is defined as follows: $\det(\bar{\partial}_a) := \otimes \det(\bar{\partial}_{k,a})$.

Definition of the Transition Functions on $\mathcal{U}_a \cap \mathcal{U}_b$. We will define how we patch together \mathcal{L}^a and \mathcal{L}^b over $\mathcal{U}^a \cap \mathcal{U}^b$ ³. On that intersection we have: $\mathcal{L}^b = \mathcal{L}^a \otimes \mathcal{L}^{a,b}$, where $\mathcal{L}^{a,b} := \otimes_k = 0^n (\det \mathcal{H}_a, b^k)^{(-1)^k}$ on $\mathcal{U}^a \cap \mathcal{U}^b$. We can identify $\mathcal{L}^{a,b}$ over $\mathcal{U}^a \cap \mathcal{U}^b$ with the line bundle bundle spanned by the section

$$\det(\bar{\partial}_{a,b}) = \otimes_{k=0}^n \det(\bar{\partial}_{k,a,b})^{(-1)^k},$$

where $\det(\bar{\partial}_{k,a,b}) := ((\bar{\partial}_{k-1, \mathcal{X}/\mathcal{M}(M)} \psi_1^{k-1}) \wedge \dots \wedge (\bar{\partial}_{k-1, \mathcal{X}/\mathcal{M}(M)} \psi_{N_k}^{k-1}) \wedge \phi_1^k \wedge \dots \wedge \phi_{M_k}^k)$, $\phi_j^k \in \text{Im } \bar{\partial}_{k-1, \mathcal{X}/\mathcal{M}(M)}$, $\psi_j^k \in \text{Im } (\bar{\partial}_{0, \mathcal{X}/\mathcal{M}(M)}^*)$, $\Delta_k \phi_j^k = \lambda_j^k \phi_j^k$, $\Delta_k (\bar{\partial}(\psi_i^{k-1})) = \lambda_i^k (\bar{\partial}(\psi_i^{k-1}))$ and $a < \lambda_i^k < b$.

Remark 13. We can view $\det(\bar{\partial}_{a,b})$ as a section of the line bundle $\mathcal{L}^{a,b}$ over $\mathcal{U}^a \cap \mathcal{U}^b$ and it defines a canonical smooth isomorphisms over $\mathcal{U}^a \cap \mathcal{U}^b$: $\mathcal{L}^a \rightarrow \mathcal{L}^a \otimes \mathcal{L}^{a,b} = \mathcal{L}^b$ ($s \rightarrow s \otimes \det(\bar{\partial}_{a,b})$).

We define the determinant line bundle \mathcal{L} by patching the trivial line bundles \mathcal{L}^a over \mathcal{U}^a by using the canonical isomorphism defined in Remark 13.

4.3. The Description of the Quillen Metric on \mathcal{L} . We now proceed to describe the Quillen metric on \mathcal{L} . Fix $a > 0$. Then the subbundles \mathcal{H}_a^k of the Hilbert bundles \mathcal{H}^k on \mathcal{U}_a inherit metrics from \mathcal{H}^k . According to standart facts from linear algebra, metrics are induced on determinants, duals, and tensor products. So the \mathcal{L}^a inherits a natural metric. We will denote by g^a the L^2 norm of the section $\det(\bar{\partial}_a)$. Clearly

$$g^a = \prod_{k=0}^n (\lambda_1^k \dots \lambda_{n_k}^k)^{(-1)^k},$$

where the product is of all non zero eigen values of the operators $\bar{\partial}_k^* \bar{\partial}_{k-1}$ which are less than a .

If $b > a$, then under the isomorphism defined in Remark 13, we have two metrics on \mathcal{L}^b and their ratio is a real number equal to the L^2 norm of the section $\|\det(\bar{\partial}_{a,b})\|^2$. The definition of the section $\det(\bar{\partial}_{a,b})$ implies that we have the following formula:

³We may suppose that $b > a$.

$$\begin{aligned} \|\det(\bar{\partial}_{a,b})\| &= \prod_{k=0}^n \prod_{i=1} \|\phi_i^k\| \prod_{j=1} \|\bar{\partial}\psi_j^k\|^{(-1)^k} = \\ \prod_{i=1} \|\phi_i^k\| \prod_{j=1} &\left\langle \bar{\partial}_k^* \bar{\partial}_{k-1} \psi_j^k, \psi_j^k \right\rangle^{(-1)^k} = \prod (\lambda_i^k)^{(-1)^k}. \end{aligned}$$

where λ_i^k are all the non-zero eigen values of the operators $\bar{\partial}_k^* \bar{\partial}_{k-1}$ such that $a < \lambda_i^k < b$. In other word, on $\mathcal{U}^a \cap \mathcal{U}^b$ $g^b = g^a \prod (\lambda_i^k)^{(-1)^k}$. To correct this discrepancy we define $\bar{g}^a = g^a \det(D^* D |_{\lambda > a})$, where

$$\det(\bar{\partial}_k^* \bar{\partial}_{k-1} |_{\lambda > a}) = -\exp(-(\zeta_k^a)'(0)) \text{ and } \zeta_k^a(s) = \sum_{\lambda_i > a} (\lambda_i^k)^s.$$

The crucial property of this regularization is that it behaves properly with respect to the finite number of eigen values, i.e.

$$\det(\bar{\partial}_k^* \bar{\partial}_{k-1} |_{\lambda > b}) = \det(\bar{\partial}_k^* \bar{\partial}_{k-1} |_{\lambda > a}) \prod_{a < i < b} \lambda_i^k$$

on the intersection $\mathcal{U}^a \cap \mathcal{U}^b$. From the last remark we deduce that \bar{g}^a and \bar{g}^b agree on $\mathcal{U}^a \cap \mathcal{U}^b$. Thus \bar{g}^a patch together to a Hermitian metric $g^{\mathcal{L}}$ on \mathcal{L} . The metric $g^{\mathcal{L}}$ will be called the Quillen metric on \mathcal{L} .

Definition 14. We will define the holomorphic analytic torsion $I(M)$, for odd dimensional CY manifold M as follows: $I(M) := \prod_{q=1}^n (\det(\Delta_q')^{(-1)^q})$. See [23].

Remark 15. It is easy to see that if $\dim_{\mathbb{C}} M = 2n$, then $\log I(M) = 0$. We know from the results in [28] that for odd dimensional CY manifolds $I(M) > 0$. So from now on we will consider only odd dimensional CY manifolds.

We will need the following result from [5] on p. 55:

Theorem 16. The Quillen norm of the C^∞ section $\det(\bar{\partial}_a)$ on \mathcal{U}_a of \mathcal{L} is equal to $I(M)$.

PROOF OF THEOREM 16. It follows from the Definition 11 of the section $\det(\bar{\partial})|_{\mathcal{U}^a}$ of \mathcal{L} and the definition of the Quillen metric that at each point $\tau \in \mathcal{M}(M)$ the following formula is true: $\|\det(\bar{\partial})_\tau|_{\mathcal{U}^a}\|_Q^2 = I(M_\tau)|_{\mathcal{U}^a}$, where $\|\det(\bar{\partial})_\tau|_{\mathcal{U}^a}\|_Q^2$ means the Quillen norm of the section $\det(\bar{\partial})_\tau|_{\mathcal{U}^a}$. Theorem 16 is proved. ■.

5. CONSTRUCTION OF A C^∞ NON VANISHING SECTION OF THE DETERMINANT LINE BUNDLE \mathcal{L} FOR ODD DIMENSIONAL CY MANIFOLDS

5.1. Some Preliminary Results. Let us denote by $\pi_* (\omega_{\mathcal{X}/\mathcal{M}(M)}) := \pi_* (\Omega_{\mathcal{X}/\mathcal{M}(M)}^{n,0})$ the relative dualizing sheaf. The local sections of $\pi_* (\omega_{\mathcal{X}/\mathcal{M}(M)})$ are families of holomorphic n -forms ω_τ on M_τ . Then on $\pi_* (\omega_{\mathcal{X}/\mathcal{M}(M)})$ we have a natural L^2 metric defined as follows:

$$\|\omega_\tau\|^2 := (-1)^{\frac{n(n-1)}{2}} (\sqrt{-1})^n \int_M \omega_\tau \wedge \bar{\omega}_\tau.$$

Theorem 17. If the dimension of the CY manifold is even then \mathcal{L} is isomorphic to the line bundle $\pi_* (\omega_{\mathcal{X}/\mathcal{M}(M)})$. If the dimension of the CY manifold is odd then the dual of \mathcal{L} is isomorphic to the line bundle $\pi_* (\omega_{\mathcal{X}/\mathcal{M}(M)})$ over $\mathcal{M}(M)$.

PROOF OF THEOREM 17. From the definition of CY says that:

$$\dim_{\mathbb{C}} H^j(M, \mathcal{O}_M) = \begin{cases} 1 & j = 0 \text{ or } j = n \\ 0 & \text{for } j \neq 0 \text{ or } n \end{cases}$$

This and the definition of CY manifold imply that

$$R^q \pi_* \mathcal{O}_M = \begin{cases} (\pi_* \omega_{\mathcal{X}/\mathcal{M}(M)})^* & j = n \\ \mathcal{O}_{\mathcal{M}(M)} & \text{for } j \neq n \end{cases}$$

From the definition of \mathcal{L} it follows that $\mathcal{L} \simeq \prod_{q=0}^n (-1)^q \det(R^q \pi_* \mathcal{O}_M)$. Combining all these equalities we deduce directly Theorem 17. \blacksquare

Corollary 18. Let M be a CY manifold of odd dimension $n = 2m + 1$. Then the index of the operator $\bar{\partial}$ on the complex defined in Definition 9 is zero.

5.2. Holomorphic Structure on the Determinant line Bundle \mathcal{L} . In [5] a canonical smooth isomorphism is constructed between the holomorphic determinant of Grothendieck-Knudsen-Mumford with Quillen determinant bundle. More precisely the following theorem is proved:

Theorem 19. Let $\pi : \mathcal{X} \rightarrow \mathcal{M}(M)$ be a holomorphic fibration with smooth fibres. Suppose \mathcal{X} admits a closed (1,1) form ψ which restricts to a Kähler metric on each fibre. Let $\mathcal{E} \rightarrow \mathcal{X}$ be a holomorphic Hermitian bundle with its Hermitian connection. Then the determinant line bundle $\mathcal{L} \rightarrow \mathcal{M}(M)$ of the relative $\bar{\partial}$ complex (coupled to \mathcal{E}) admits a holomorphic structure. The canonical connection (constructed in [3]) on \mathcal{L} is the Hermitian connection for the Quillen metric. Finally, if the index of $\bar{\partial}$ is zero, the section $\det(\bar{\partial}_E)$ of \mathcal{L} is holomorphic.

From now on we will consider the family of CY manifolds $\mathcal{X} \rightarrow \mathcal{M}(M)$ as defined in Theorem 8 together with the trivial line bundle \mathcal{E} over $\mathcal{M}(M)$. It is easy to see that the family $\mathcal{X} \rightarrow \mathcal{M}(M)$ fulfill the conditions of the Theorem 19.

5.3. Construction of a C^∞ Section of the Determinant Line Bundle \mathcal{L} with Quillen Norm Ray-Singer Equal to Ray-Singer Analytic Torsion.

Definition 20. Let $\mathcal{H}_+ = \bigoplus_k L^2(M, \Omega_M^{0,2k})$ and $\mathcal{H}_- = \bigoplus_k L^2(M, \Omega_M^{0,2k+1})$ and $D = \bar{\partial}_{\mathcal{X}/\mathcal{M}(M)} + \bar{\partial}_{\mathcal{X}/\mathcal{M}(M)}^*$.

Theorem 21. Let M be a CY manifold of odd dimension, then as C^∞ bundle the determinant line bundle \mathcal{L} is trivial and there exists a global C^∞ section $\det(\bar{\partial})$ of $\mathcal{L} \rightarrow \mathcal{M}(M)$, which has no zeroes on $\mathcal{M}(M)$ and whose Quillen norm is the Ray Singer Analytic Torsion.

PROOF OF THEOREM 21. **PROOF:** The proof of Theorem 21 is based on the following three Theorems:

Theorem 22. *The first Chern class of the relative dualizing sheaf $\pi_*\omega_{\mathcal{X}/\mathcal{M}(M)}$ is given locally by the formula $c_1(\pi_*\omega_{\mathcal{X}/\mathcal{M}(M)}) = dd^c(\langle \omega_\tau, \omega_\tau \rangle) = -\text{Im}(\text{Weil-Petersson metric})$, where ω_τ is a holomorphic family of holomorphic n forms.. (See [26].)*

Theorem 23. *Locally on the moduli space $\mathcal{M}(M)$ the following formula holds*

$$dd^c(\log(\det(\Delta_0))) = -\text{Im}(\text{Weil-Petersson metric}). \text{ (See [28].)}$$

Theorem 24. **i.** $I(M) = (\det(\Delta_0))^2$ **ii.** around each point $\tau \in \mathcal{U}_\tau \in \mathcal{M}(M)$ we have $I(M)|_{\mathcal{U}_\tau} = \langle \omega, \omega \rangle |\phi|^2$, where ϕ is a holomorphic function on \mathcal{U} and **iii.** The positive function $\det(\Delta_0)$ is bounded by a constant C , i.e. $\det(\Delta_0) < C$. (See [28].)

We will proof the following Lemma:

Lemma 25. *The first Chern class $c_1(\mathcal{L})$ of the C^∞ determinant line bundle \mathcal{L} is equal to zero in $H^2(\mathcal{M}(M), \mathbb{Z})$.*

Proof of Lemma 25: Theorem 17 implies that when the CY manifold M has an odd dimension then the determinant line bundle \mathcal{L} is isomorphic to the relative dualizing sheaf $\pi_*(\omega_{\mathcal{X}/\mathcal{M}(M)})$. So we need to proof that $c_1(\pi_*(\omega_{\mathcal{X}/\mathcal{M}(M)})) = 0$. The proof of Lemma 25 is based on the following obeservation: Notice that the definition of the Ray Singer analytic torsion implies that $I(M_\tau)$ is a positive function different from a constant on $\mathcal{M}(M)$. From Theorem 24 we know that we have the following local expression of $I(M_\tau)$:

$$I(M_\tau)|_{\mathcal{U}} = \|\omega_\tau\|^2 |\phi|^2$$

where ω_τ is a holomorphic family of holomorphic n -forms on M_τ , ϕ is a holomorphic function on \mathcal{U} and

$$\|\omega_\tau\|^2 = (\sqrt[n]{-1})^n (-1)^{\frac{n(n-1)}{2}} \int_{M_\tau} \omega_\tau \wedge \overline{\omega_\tau}.$$

Theorems 23 and 22 imply that

$$\frac{\sqrt[n]{-1}}{2} \partial \overline{\partial} \log(I(M_\tau)) = c_1(\mathcal{L}) = c_1(\pi_*(\omega_{\mathcal{X}/\mathcal{M}(M)})).$$

Since $\partial \overline{\partial} \log(I(M_\tau)) = d(\overline{\partial} \log(I(M_\tau)))$ we deduce that

$$\frac{\sqrt[n]{-1}}{2} \partial \overline{\partial} \log(I(M_\tau)) = c_1(\mathcal{L}) = \frac{\sqrt[n]{-1}}{2} d(\overline{\partial} \log(I(M_\tau))) = d\alpha.$$

So $c_1(\mathcal{L}) = 0$ in $H^2(\mathcal{M}(M), \mathbb{Z})$. This proves Lemma 25. \blacksquare .

Corollary 26. *The determinant line bundle \mathcal{L} as a C^∞ bundle is trivial.*

So the first part of Theorem 21 is proved. The proof of the second part of Theorem 21 is based on the following Lemma:

Lemma 27. *There exists a non vanishing global section $\det(\bar{\partial})$ of the dual of the determinant line bundle \mathcal{L} such that the Quillen norm of $\det(\bar{\partial}) = I(M)$.*

Proof of Lemma 27: From Corollary 26 we can conclude the existence of a global C^∞ section ω of $\mathcal{L} \rightarrow \mathcal{M}(M)$, which has no zeroes on $\mathcal{M}(M)$ and such that for each $\tau \in \mathcal{M}(M)$ it has L^2 norms 1, i.e. we have $\|\omega_\tau\|^2 = 1$. Since M_τ is an odd dimensional CY manifold we know from Theorem 17 that \mathcal{L} is isomorphic to $\pi_*(\omega_{\mathcal{X}/\mathcal{M}(M)})$. The non vanishing section ω of the determinant line bundle \mathcal{L} can be interpreted as a family of $(2n+1,0)$ forms ω which generate the kernel of $D^* : \mathcal{H}_- \rightarrow \mathcal{H}_+$. The kernel of $D : \mathcal{H}_+ \rightarrow \mathcal{H}_-$ is generated by the constant 1. This follows directly from the definition of the CY manifold.

From Definition 11 of the section $\det(\bar{\partial}_a)$ on the open set $\mathcal{U}_a \subset \mathcal{M}(M)$, the existence of a C^∞ family of antiholomorphic forms ω_τ with L^2 norm 1, which trivializes $R^{2n+1}\pi_*(\mathcal{O}_{\mathcal{X}})$ over $\mathcal{M}(M)$ and the definition of the transition functions $\{\sigma_{a,b}\}$ of \mathcal{L} with respect to the covering $\{\mathcal{U}_a\}$, we deduce that for $b > a$ we have on $\mathcal{U}_a \cap \mathcal{U}_b$ $\det(\bar{\partial}_b) = \det(\bar{\partial}_a)\sigma_{a,b}$. This fact and Theorem 16 imply that we constructed a global C^∞ section $\det(\bar{\partial})$ of \mathcal{L} whose Quillen norm is the Ray Singer Analytic Torsion. So the determinant line bundle \mathcal{L} is a trivial C^∞ line bundle. Theorem 21 is proved. ■.

Corollary 28. *The determinant line bundle as a holomorphic bundle is flat over $\mathcal{M}(M)$.*

6. THE ANALOGUE OF THE DEDEKIND ETA FUNCTION FOR ODD DIMENSIONAL CY MANIFOLDS

6.1. Metrics on Vector Bundles with Logarithmic Growth. In Theorem 7 we constructed the moduli space $\mathcal{M}(M)$ of CY manifolds. From the results in [30] and Theorem 7 we know that $\mathcal{M}(M)$ is a quasi-projective non-singular variety. Using Hironaka's resolution theorem, we may suppose that $\mathcal{M}(M) \subset \overline{\mathcal{M}(M)}$, where $\overline{\mathcal{M}(M)} \setminus \mathcal{M}(M) = \mathcal{D}_\infty$ is a divisor with normal crossings. We need to show, now how we will extend the determinant line bundle \mathcal{L} to a line bundle $\overline{\mathcal{L}}$ to $\overline{\mathcal{M}(M)}$. For this reason we are going to recall the following definitions and results from [21]. We will look at the polycylinders $D^N \subset \overline{\mathcal{M}(M)}$, where D is the unit disk and $N = \dim \overline{\mathcal{M}(M)}$ and

$$D^N \cap \mathcal{D}_\infty = \{\text{union of hyperplanes; } \tau_1 = 0, \dots, \tau_k = 0\}.$$

Hence $D^N \cap \mathcal{M}(M) = (D^*)^k \times D^{N-k}$. In D^* we have the Poincare metric

$$ds^2 = \frac{|dz|^2}{|z|^2(\log|z|)^2}$$

and in D we have the simple metric $|dz|^2$, giving us a product metric on $(D^*)^k \times D^{N-k}$ which we call $\omega^{(P)}$.

A complex-valued C^∞ p-form η on $\mathcal{M}(M)$ is said to have Poincare growth on $\overline{\mathcal{M}(M)} \setminus \mathcal{M}(M)$ if there is a set of *if polycylinders* $\mathcal{U}_\alpha \subset \overline{\mathcal{M}(M)}$ covering $\overline{\mathcal{M}(M)} \setminus \mathcal{M}(M)$ such that in each \mathcal{U}_α an estimate of the following type holds:

$$|\eta(\tau_1, \dots, \tau_N)| \leq C_\alpha \omega_{\mathcal{U}_\alpha}^{(P)}(\tau_1, \overline{\tau_1}) \dots \omega_{\mathcal{U}_\alpha}^{(P)}(\tau_N, \overline{\tau_N}).$$

This property is independent of the covering \mathcal{U}_α of $\overline{\mathcal{M}(M)} \setminus \mathcal{M}(M)$ but depends on the compactification $\overline{\mathcal{M}(M)}$. If η_1 and η_2 both have Poincare growth on $\overline{\mathcal{M}(M)} \setminus \mathcal{M}(M)$, then so does $\eta_1 \wedge \eta_2$. The basic property of the Poincare growth is the following:

Theorem 29. A p -form η with a Poincare growth on $\overline{\mathcal{M}(M)} \setminus \mathcal{M}(M)$, has the property that for every C^∞ (r-p) form ψ on $\overline{\mathcal{M}(M)}$ we have:

$$\int_{\overline{\mathcal{M}(M)} \setminus \mathcal{M}(M)} |\eta \wedge \psi| < \infty.$$

Hence, η defines a current $[\eta]$ on $\overline{\mathcal{M}(M)}$.

PROOF OF THEOREM 29. For the proof see [21]. ■.

A complex valued C^∞ p-form η on $\overline{\mathcal{M}(M)}$ is good on M if both η and $d\eta$ have Poincare growth. Let \mathcal{E} be a vector bundle on $\overline{\mathcal{M}(M)}$ with a Hermitian metric h . We will call h a good metric on $\overline{\mathcal{M}(M)}$ if the following holds: **i.** for all $x \in \overline{\mathcal{M}(M)} \setminus \mathcal{M}(M)$, there exists sections e_1, \dots, e_m of \mathcal{E} which form a basis of $\mathcal{E}|_{D^r \setminus (D^r \cap \mathcal{D}_\infty)}$. **ii.** In a neighborhood D^r of x in which $\overline{\mathcal{M}(M)} \setminus \mathcal{M}(M)$ is given by $z_1 \times \dots \times z_k = 0$. **iii.** The metric $h_{ij} = h(e_i, e_j)$ has the following properties: **a.** $|h_{ij}|, (\det(h))^{-1} \leq C \left(\sum_{i=1}^k \log |z_i| \right)^{2m}$, for some $C > 0$, $m \geq 0$. **b.** The 1-forms $((dh)h^{-1})_{ij}$ are good forms on $\overline{\mathcal{M}(M)} \cap D^r$.

It is easy to prove that there exists a unique extension $\overline{\mathcal{E}}$ of \mathcal{E} on $\overline{\mathcal{M}(M)}$, i.e. $\overline{\mathcal{E}}$ is defined locally as holomorphic sections of \mathcal{E} which have finite norm in h .

Theorem 30. Let (\mathcal{E}, h) be a vector bundle with a good metric on $\mathcal{M}(M)$, then the Chern classes $c_k(\mathcal{E}, h)$ are good forms on $\overline{\mathcal{M}(M)}$ and the currents $[c_k(\mathcal{E}, h)]$ represent the cohomology classes

$$c_k(\mathcal{E}, h) \in H^{2k}(\overline{\mathcal{M}(M)}, \mathbb{Z}).$$

PROOF OF THEOREM 30. For the proof see [21]. ■.

6.2. Applications of Mumford's Results to the Moduli of Odd Dimensional CY.

We are going to prove the following result:

Theorem 31. Let $\pi : \mathcal{X} \rightarrow \mathcal{M}(M)$ be the flat family of non-singular CY manifolds. Let $\pi_*(\Omega_{\mathcal{X}/\mathcal{M}(M)}^{n,0})$ be equipped with the metric h defined as follows:

$$h(\omega_\tau, \omega_\tau) = (-1)^{\frac{n(n-1)}{2}} (\sqrt{-1})^n \int_M \omega_\tau \wedge \overline{\omega_\tau}.$$

Then h is a good metric.

PROOF OF THEOREM 31. Let $\tau_0 \in \mathcal{D}_\infty$ and let D be a one dimensional disk in $\overline{\mathcal{M}(M)}$ which intersects \mathcal{D}_∞ on τ_0 and $D^* = D \setminus \tau_0 \in \mathcal{M}(M)$. Over $D \setminus \tau_0$ we have a family $\mathcal{M}_{D^*} \rightarrow D \setminus \tau_0$ of CY manifolds. We will assume that D is the unit disk and τ_0 is the origin of the disk. We know from the theory of Hodge structures that if $\{\gamma_1, \dots, \gamma_{b_n}\}$ is a basis of $H^n(M, \mathbb{Z})/\text{Tor}$ then the functions:

$$(\dots, \int_{\gamma_i} \omega_\tau, \dots)$$

for $0 < |\tau| < 1$ and $0 < \arg(\tau) < 2\pi$ are solutions of differential equation with regular singularities. From the fact that the solutions of any differential equation with regular singularities has a logarithmic growth and

$$h(\omega_\tau, \omega_\tau) = \left(\dots, \int_{\gamma_i} \omega_\tau, \dots \right) (\langle \gamma_i, \gamma_j \rangle) \left(\dots, \int_{\gamma_i} \overline{\omega_\tau}, \dots \right)^t$$

we deduce that

$$h(\omega_\tau, \omega_\tau) \leq C \left(\sum_{i=1}^k \log |\tau_i| \right)^{2m}.$$

From here we conclude that the form $\partial(\log(h))$ has also a logarithmic growth. Our theorem is proved. ■.

6.3. Construction of the Analogue of the Dedekind η Function for Odd Dimensional CY manifolds.

Theorem 32. *Let M be an odd dimensional CY manifold, then the C^∞ section $\det(\bar{\partial})$ of the determinant line bundle \mathcal{L} constructed in Theorem 21 can be prolonged to a section $\overline{\det(\bar{\partial})}$ of the line bundle $\overline{\mathcal{L}}$ and $\det(\bar{\partial})$ vanishes on the discriminant locus $\mathcal{D}_\infty := \overline{\mathcal{M}(M)} \setminus \mathcal{M}(M)$.*

PROOF OF THEOREM 32. We know from Theorem 21 that the Quillen norm of the section $\det(\bar{\partial})$ of the determinant line bundle \mathcal{L} on $\mathcal{M}(M)$ is equal to the holomorphic Ray-Singer analytic torsion $I(M)$, i.e. $\|\det(\bar{\partial})\|_Q^2 = I(M)$. On the other hand we proved in [28] that $I(M) = (\det(\Delta_0))^2$. This fact and Theorems 23 and 22 imply that locally on $\mathcal{M}(M)$ the following formula is true:

$$dd^c \left(\log \left(\frac{I(M)}{\langle \omega_\tau, \omega_\tau \rangle} \right) \right) = dd^c \left(\log \left(\frac{\det(\Delta_0)}{\langle \omega_\tau, \omega_\tau \rangle} \right) \right) = 0.$$

From here we deduce in [28] that for each point $\tau \in \mathcal{M}(M)$ there exists an open set $\tau \in \mathcal{U}_\tau$ such that we have

$$\|\det(\bar{\partial})|_{\mathcal{U}_\tau}\|_Q^2 = \langle \omega_\tau, \omega_\tau \rangle |f(\tau)|^2,$$

where f is a holomorphic function in \mathcal{U}_τ . Let us choose \mathcal{U} such that $\mathcal{U} \cap \overline{\mathcal{M}(M)} \neq \emptyset$. Let $\tau_0 \in \mathcal{D}_\infty \cap \mathcal{U}$. We will prove that we can continue f locally around a point $\tau_0 \in \mathcal{D}_\infty = \overline{\mathcal{M}(M)} \setminus \mathcal{M}(M)$. Indeed we proved in Theorem 31 that $\langle \omega_\tau, \omega_\tau \rangle$ have a logarithmic growth around $\tau_0 \in \mathcal{D}_\infty = \overline{\mathcal{M}(M)} \setminus \mathcal{M}(M)$. Theorem 24 implies that $\|\det(\bar{\partial})\|_Q^2 = I(M) = (\det(\Delta_0))^2 \leq C$ and $I(M)|_{\mathcal{U}_\tau} = \langle \omega, \omega \rangle |f|^2$ so we can conclude that $\lim_{\tau \rightarrow \tau_0} f(\tau) = 0$ since $\lim_{\tau \rightarrow \tau_0} \langle \omega_\tau, \omega_\tau \rangle = \infty$ and $\langle \omega_\tau, \omega_\tau \rangle$ has a logarithmic growth. From here we deduce, and f can be continued around any point $\tau \in \mathcal{D}_\infty = \overline{\mathcal{M}(M)} \setminus \mathcal{M}(M)$. This fact and Theorem 31 implies that the canonical section $\det(\bar{\partial})|_{\mathcal{U}}$ can be prolonged to a section $\overline{\det(\bar{\partial})}_{\mathcal{U}}$ of the line bundle $\overline{\mathcal{L}}$ and $\det(\bar{\partial})$ vanishes on the discriminant locus $\mathcal{D}_\infty := \overline{\mathcal{M}(M)} \setminus \mathcal{M}(M)$. Theorem 32 is proved. ■.

Theorem 33. *There exists a multi valued holomorphic section η of the dual of the determinant line bundle \mathcal{L} over $\mathcal{M}(M)$ such that the norm of η with respect to the metric defined in Theorem 31 is equal to the Ray Singer Analytic torsion $I(M) = \det(\Delta_0)^2$.*

PROOF OF THEOREM 33. Theorem 21 implies that the line bundle \mathcal{L} is a trivial C^∞ bundle. So the pullback $\pi^*(\mathcal{L})$ on the universal cover $\widetilde{\mathcal{M}(M)}$ of $\mathcal{M}(M)$ will be a trivial line bundle. Let $\tilde{\sigma}$ be any non vanishing section of $\pi^*(\mathcal{L})$. Then since $\mathcal{L} = \widetilde{\mathcal{M}(M)} \times \mathbb{C}/\sim$, where $(\tau, t) \sim (\tau_1, t_1) \Leftrightarrow \tau_1 = g\tau$ and $t_1 = \chi(g)t$, where $g \in \pi_1(\mathcal{M}(M))$ and χ is a character of $\pi_1(\mathcal{M}(M))$ that defines \mathcal{L} . Theorem 33 is proved. \blacksquare .

We know from the results in [5] and 16 that the determinant line bundle \mathcal{L} over $\mathcal{M}(M)$ has a holomorphic structure. In the next Theorem we will consider \mathcal{L} as a holomorphic line bundle defined over $\mathcal{M}(M)$.

Theorem 34. There exists a positive integer N such that $\mathcal{L}^{\otimes N}$ is a trivial complex analytic line bundle over $\mathcal{M}(M)$.

PROOF OF THEOREM 34. According to Theorem 17 $\mathcal{L} \simeq R^0\pi_*(\Omega_{\mathcal{X}/\mathcal{M}(M)}^{2n+1,0})$, where $\dim_{\mathbb{C}} M = 2n + 1$. Therefore \mathcal{L} is a subbundle of the flat vector bundle $R^{2n+1}\pi_*\mathbb{C}$ where \mathbb{C} is the locally constant sheaf on \mathcal{X} , and $\pi : \mathcal{X} \rightarrow \mathcal{M}(M)$ is the versal family of CY manifolds over $\mathcal{M}(M)$. We know from Theorem 7 that $\mathcal{M}(M) = \mathcal{T}(M)/\Gamma$, where $\mathcal{T}(M)$ is the Teichmüller space and Γ is a subgroup of the mapping class group of M and according to [24] Γ is an arithmetic group.

If we lift the flat bundle $R^n\pi_*\mathbb{C}$ on $\mathcal{T}(M)$, then $R^{2n+1}\pi_*\mathbb{C}$ will be a trivial bundle isomorphic to $\mathcal{T}(M) \times H^{2n+1}(M_0, \mathbb{C})$. Let us denote by

$$\sigma : \mathcal{T}(M) \rightarrow \mathcal{M}(M) = \mathcal{T}(M)/\Gamma$$

the natural map. Clearly $\sigma^*(\mathcal{L})$ will be a flat complex analytic subbundle of the trivial bundle $\mathcal{T}(M) \times H^{2n+1}(M_0, \mathbb{C})$.

Proposition 35. Let N be a quasi-projective variety, $\mathcal{E} \cong \mathbb{C}^n \times N$ be a trivial bundle and \mathcal{L} be a flat line bundle over N such that $\mathcal{L}^* \subset \mathcal{E}$, then \mathcal{L} is also trivial.

Proof of Proposition 35: Let \tilde{N} be the universal cover of N . Clearly $\pi_1(N)$ acts without fixed points on \tilde{N} . The pullback of \mathcal{L} on \tilde{N} will be denoted by $\tilde{\mathcal{L}}$. $\tilde{\mathcal{L}}$ will be a trivial line bundle since \mathcal{L}^* is a flat bundle over N . Let us denote by $\tilde{\mathcal{E}}$ the pullback of \mathcal{E} on \tilde{N} .

$\pi_1(N)$ acts trivially on the trivial vector bundle $\tilde{\mathcal{E}}$ since \mathcal{E} is a trivial vector bundle on N . The condition $\mathcal{L} \subset \mathcal{E}$ implies that $\tilde{\mathcal{L}} \subset \tilde{\mathcal{E}}$. $\pi_1(N)$ acts trivially on the trivial line bundle $\tilde{\mathcal{L}}$ since $\pi_1(N)$ acts trivially on the trivial vector bundle $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{L}} \subset \tilde{\mathcal{E}}$. \mathcal{L} is a trivial bundle over N since $\pi_1(N)$ acts trivially on the trivial line bundle $\tilde{\mathcal{L}}$. Proposition 35 is proved. \blacksquare .

Proposition 35 implies we that $\sigma^*(\mathcal{L})$ will be a trivial line bundle. The fact that $\sigma^*(\mathcal{L}) \cong \mathbb{C} \times \mathcal{T}(M)$ is a trivial bundle implies that $\mathcal{L} \cong \mathbb{C} \times \mathcal{T}(M)/\Gamma$, where Γ acts in a natural way on the Teichmüller space and it acts by a character $\chi \in \text{Hom}(\Gamma, \mathbb{C}_1^*) \cong \text{Hom}(\Gamma/\Gamma, \mathbb{C}_1^*)$ on \mathbb{C} . According to a Theorem of Kazhdan Γ/Γ is a finite group since Γ is an arithmetic group of rank ≥ 2 according to [24] and [8]. From here we deduce that \mathcal{L}^N will be a trivial bundle on $\mathcal{M}(M)$, where $N = \#\Gamma/\Gamma$. Theorem 34 is proved. \blacksquare .

Corollary 36. There exists a holomorphic section η^N of the trivial bundle $(\mathcal{L}^*)^N$ such that it can be prolonged to a holomorphic section $\overline{\eta^N}$ of $(\overline{\mathcal{L}^*})^N$ whose zero set is supported by \mathcal{D}_∞ and the Quillen norm $\|\eta^N\|_Q^2 = \det(\Delta_0)^{2N}$.

Proof of Corollary 36: As we pointed out we can prolong the dual of the determinant line bundle \mathcal{L} from $\mathcal{M}(M)$ to a holomorphic line bundle $\overline{\mathcal{L}}^*$ over $\overline{\mathcal{M}(M)}$. In Theorem 21 we constructed a non vanishing C^∞ section $\det(\overline{\partial})$ of \mathcal{L}^* over $\mathcal{M}(M)$. We know that the norm of $\det(\overline{\partial})$ with respect of the metric on \mathcal{L} defined in Theorem 31 is exactly equal to the Ray Singer Analytic Torsion $I(M) = \det(\Delta_0)^2$. From Theorem 32 we know that we can prolong the section $\det(\overline{\partial})$ to a section $\det(\overline{\partial})$ of the line bundle $\overline{\mathcal{L}}^*$ over $\overline{\mathcal{M}(M)}$ and the support of the zero set of $\det(\overline{\partial})$ is exactly \mathcal{D}_∞ . From here we deduce that the Poincare dual homology class of the Chern class of $\overline{\mathcal{L}}$ is an effective divisor whose support is the same as that of \mathcal{D}_∞ . Combining this fact together with Theorem 34 we conclude that there exists a holomorphic section $\overline{\eta^N}$ of the line bundle $(\overline{\mathcal{L}}^*)^N$ over $\overline{\mathcal{M}(M)}$ whose zero set is supported by \mathcal{D}_∞ and the multiplicities of the components of the irreducible divisors of $(\overline{\eta})^N$ are the same as of $(\det(\overline{\partial}))^N$. The fact that η^N is defined up to a constant and Corollary 33 we conclude that after normalizing η^N its Quillen norm $\|\eta^N\|_Q^2$ will be $\det(\Delta_0)^{2N}$. Corollary 36 is proved. ■

7. SOME PROBLEMS

Let us define $Sh(C, E, Z)$ to be the set of all families $\pi : Y \rightarrow C$ of fixed type CY manifolds Z over a fix complete algebraic curve C with a fixed set of points over which the fibres are singular, up to isomorphisms.

Problem 37. *Is the set $Sh(C, E, Z)$ finite?*

The results of this paper combined with the results from [17] imply that the set $Sh(C, E, Z)$ is discrete. In order to prove that it is finite one need to find a uniform bound on the volume of the image of the curve in the moduli space $\mathcal{M}(M)$ of CY manifolds. We found a bound of the images of fix C and fix set of points E on C over which the fibres are singular by using Gauss-Bonnet theorem and the fact that Weil-Petersson metric is complete on the moduli space of pseudo polarized algebraic K3 surfaces. This method does not work for CY threefolds, since Weil-Petersson metric is not complete.

Problem 38. *Can one find a product formulas for η around points of maximal degenerations, which means that around such points the monodromy operator has index of unipotency $n + 1$?*

For more precise discussion of Problem 38 see [29]. Problem 38 is closely related to the paper [2] and more precisely to the counting problem of elliptic curves on CY threefold.

Problem 39. *Is it true that any CY manifold can be deformed to an algebraic manifold with one conic singularity?*

Problem 39 is related to the following problem: Let M be an algebraic variety embedded in \mathbb{P}^N . Suppose that the component of the Hilbert scheme $\mathcal{H}'_{M/\mathbb{P}^N}$ that contains an open non singular quasi-projective subscheme $\mathcal{H}_{M/\mathbb{P}^N}$. Let \mathcal{D}_M be the set of the points in $\mathcal{H}_{M/\mathbb{P}^N}$ that corresponds to singular varieties. It is not difficult to prove that \mathcal{D}_M is a closed subvariety in $\mathcal{H}_{M/\mathbb{P}^N}$. Suppose that \mathcal{D}_M is a divisor in $\mathcal{H}_{M/\mathbb{P}^N}$.

Problem 40. *Is it true that the generic point of \mathcal{D}_M corresponds to a projective manifold with only one conic singularity?*

The results of this paper suggest that one can expect that the Hilbert scheme $\mathcal{H}'_{M/\mathbb{P}^N}$ of odd dimensional CY manifold M contains a non singular open quasi-projective variety $\mathcal{H}_{M/\mathbb{P}^N}$ such that the discriminant locus \mathcal{D}_M is a divisor.

Problem 40 is closely related to the Miles Ried's conjecture that the moduli of all CY threefolds is connected.

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